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BENDING AND TORSION OF CANTILEVERS WITH CROSS-SECTIONS IN THE FORM OF SOLID CIRCULAR SECTORS

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Abstract—Solutions for the problems of bending and torsion of cantilever beams with solid circular sector sections are obtained in closed form. It is assumed that the circular sectors are confined in angles π, $\pi/2N$ in which N is an integer. It is also assumed that all conditions needed for the St. Venant's solution of flexure of a cantilever beam are met. The method of solution is as follows. First, the solution for flexure of a semicircular cross-section beam, with the load in its plane of symmetry, is obtained in closed form. Next, considering one-half of this semicircular section, a twisting moment is superposed in order to eliminate the rotation created by the flexure. Thus, the closed form solution for the bending of a quarter of a circle cross-section beam is obtained. This procedure is continued in order to obtain solutions for bending of beams having circular sector cross-sections of angles $\pi/4$, $\pi/8$, ... In these steps, closed form solutions for torsion of these sector sections are utilized. Numerical results for the cases of sectors with angles π, $\pi/2$ and $\pi/4$ are presented.

INTRODUCTION

Analytical solutions for flexure and torsion of bars according to St. Venant's theory have been obtained in a few investigations. Seegar and Pearson (1920) derived solutions for the problem of bending of beams with circular sector cross-sections. Later Timoshenko and Goodier (1951) gave the solution for the torsion of circular sector cross-section bars. However, these authors obtained their solutions in the form of infinite series. Greenhill (1878) also gave a solution for the torsion of a sector of a circle in closed form involving two integrals. As cited by Love (1944), these integrals can be evaluated for certain values of the angle of the sector. Recently, Naghdi (1985) obtained a closed form solution for the torsion of a bar with semicircular cross-section. Using polar coordinates, he first derived the results in the form of infinite series. Then, reducing these series to certain forms and employing some known formulas (Gradshteyn and Ryzhik, 1965; Kantorovich and Krylov, 1964), he obtained the closed form solution. A technique similar to that of Naghdi (1985) will be employed here in order to obtain the closed form solution for the flexure of a semicircular cross-section cantilever with the load in its plane of symmetry. The readers are reminded here that the solution for flexure of a semicircular section beam subject to a load perpendicular to its plane of symmetry was given in Timoshenko and Goodier (1951) and Sokolnikoff (1956). This solution, along with the results obtained in this investigation and those of the work of Naghdi (1985), complete the solution for unsymmetrical bending, combined with torsion, of a semicircular section beam.

Closed form solutions for the torsion of bars with circular sector sections of angles $\pi/2$, $\pi/4$, etc. are derived here. These solutions are needed, as explained in the following. If we consider the displacement field in one-half of the section in the flexure of a semicircular section beam loaded in the plane of symmetry, we see that there exists a rotation due to flexure. In order to eliminate this rotation, we can superpose the solution for torsion of the quarter of a circle section bar. The procedure can be continued in order to obtain the solutions for unsymmetrical flexure, combined with torsion, of cantilevers with circular sector sections of angles $\pi/4$, $\pi/8$, etc.

Various numerical results for these beams with sector of angles π, $\pi/2$ and $\pi/4$ are presented.

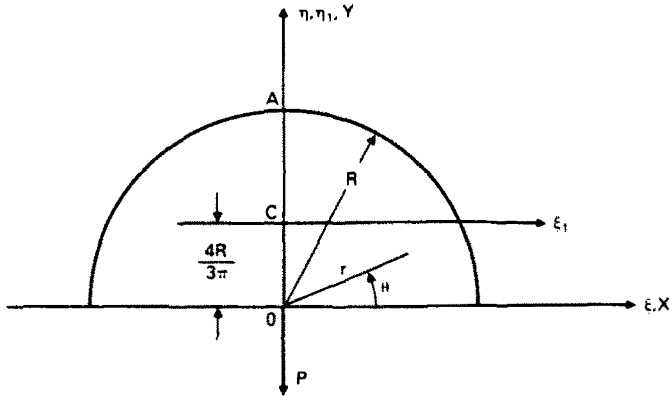

Fig. 1. Semicircular cross-section beam subject to a load in its plane of symmetry.

METHOD OF SOLUTION

(a) *Solution for the flexure of a semicircular cross-section cantilever*

Consider a prismatic cantilever beam whose cross-section is a solid semicircle, as shown in Fig. 1. The nondimensional rectangular coordinates $\xi = X/R$, $\eta = Y/R$, as well as the dimensionless polar coordinates $\rho = r/R$, and θ centered at 0, are chosen for the analysis. In addition, the centroidal axes ξ_1 and η_1 are also utilized. According to St. Venant's theory of bending of beams (Sokolnikoff, 1956), the equation

$$\nabla^2\phi_2 = 0, \quad \nabla^2 \equiv \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}, \tag{1}$$

in which ϕ_2 is the nondimensional stress function, must be satisfied, and the condition

$$\frac{\partial\phi_2}{\partial\eta} = [(1+\nu)\eta_1^2 - \nu\xi_1^2] \cos(\eta_1, n) \tag{2}$$

has to be fulfilled on the boundary of the semicircular region. In (2), n is the outward normal direction to the boundary, and ν is Poisson's ratio. At the semicircular and diametral boundaries $\rho = 1$, and $\eta = 0$, the boundary condition (2) can be written in terms of θ and ξ:

$$\left.\frac{\partial\phi_2}{\partial n}\right|_{\rho=1} = \left[(1+\nu)\left(\sin^2\theta - \frac{8}{3\pi}\sin\theta + \frac{16}{9\pi^2}\right) - \nu\cos^2\theta\right]\sin\theta, \tag{3}$$

$$\left.\frac{\partial\phi_2}{\partial n}\right|_{\eta=0} = -\left[(1+\nu)\frac{16}{9\pi^2} - \nu\xi^2\right]. \tag{4}$$

Let us now introduce another harmonic function Φ such that:

$$\Phi = \phi_2 - \frac{\nu}{3}(\eta^3 - 3\eta\xi^2) + \frac{4}{3\pi}(1+\nu)(\eta^2 - \xi^2) - \frac{16(1+\nu)}{9\pi^2}\eta. \tag{5}$$

The differential eqn (1) and the boundary conditions (3) and (4) then become:

$$\nabla^2\Phi = 0, \tag{6}$$

$$\left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=1} = (\tfrac{1}{2}\nu - \tfrac{1}{4})\sin 3\theta + (\tfrac{1}{2}\nu + \tfrac{3}{4})\sin\theta - (1+\nu)\left(\frac{4}{3\pi}\right) - (1+\nu)\left(\frac{4}{3\pi}\right)\cos 2\theta, \tag{7}$$

$$\left.\frac{\partial\Phi}{\partial\eta}\right|_{\eta=0} = \left.\frac{\partial\Phi}{\partial\theta}\right|_{\eta=0} = 0. \tag{8}$$

The solution for the harmonic function Φ is written in polar coordinates in the form

$$\Phi = C_2\rho^2 \cos 2\theta + \sum_{m=1}^{\infty} A_m\rho^{2m} \cos 2m\theta. \tag{9}$$

It is seen that with this choice of Φ, the boundary condition (8) is automatically satisfied.

In order to satisfy condition (7), we expand $\sin\theta$ and $\sin 3\theta$ in that relation in terms of Fourier series in $\cos 2m\theta$. Thus, condition (7) becomes

$$\left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=1} = -\left(\frac{4}{3\pi}\right)(1+\nu)\cos 2\theta + \frac{2}{\pi}(\tfrac{1}{2}\nu-\tfrac{1}{4})\sum_{m=1}^{\infty}\left(\frac{1}{3+2m}+\frac{1}{3-2m}\right)\cos 2m\theta$$
$$+\frac{2}{\pi}(\tfrac{1}{2}\nu+\tfrac{3}{4})\sum_{m=1}^{\infty}\left(\frac{1}{1+2m}+\frac{1}{1-2m}\right)\cos 2m\theta. \tag{10}$$

Substituting expression (9) into (10) and setting the coefficients of $\cos 2m\theta$ equal on both sides, we find the unknown constants C_2 and A_m:

$$C_2 = -\left(\frac{2}{3\pi}\right)(1+\nu), \tag{11}$$

$$A_m = \frac{1}{\pi}(\nu-\tfrac{1}{2})\left[\frac{1}{2m(3+2m)}+\frac{1}{2m(3-2m)}\right]+\frac{1}{\pi}(\nu+\tfrac{3}{2})\left[\frac{1}{2m(1+2m)}+\frac{1}{2m(1-2m)}\right]. \tag{12}$$

Substituting (11) and (12) into (9), we obtain the solution in the form of an infinite series. We now employ partial fraction techniques in order to write the terms in expression for A_m in simpler forms, and use the known identities (Naghdi, 1985; Gradshteyn and Ryzhik, 1965; Kantorovich and Krylov, 1964):

$$F_1(\rho,\theta) = \sum_{m=1,3,5}^{\infty}\frac{\rho^m}{m}\cos m\theta \equiv \frac{1}{4}\ln\left(\frac{\cosh\lambda+\cos\theta}{\cosh\lambda-\cos\theta}\right),$$

$$F_2(\rho,\theta) = \sum_{m=1,3,5}^{\infty}\frac{\rho^m}{m}\sin m\theta \equiv \frac{1}{2}\left\{-\frac{\pi}{2}+\arctan[G(\lambda,\theta)]+\arctan[G(\lambda,\pi-\theta)]\right\},$$

$$F_3(\rho,\theta) = \sum_{m=2,4,6}^{\infty}\frac{\rho^m}{m}\cos m\theta \equiv \frac{1}{4}\ln\left[\frac{(\cosh\lambda-1)^2}{\cosh^2\lambda-\cos^2\theta}\right]-\ln(1-e^{-\lambda}),$$

$$\lambda = -\ln\rho,\quad G(\lambda,\theta) = \frac{(1+\cosh\lambda)\tan(\theta/2)}{\sinh\lambda},$$

$$1>\rho>0,\quad \theta\neq 0,\pi, \tag{13}$$

and similar formulas in order to write Φ and its derivatives in closed form. For example, we have:

$$\Phi = C_2\rho^2 \cos 2\theta + \frac{1}{\pi}(\nu - \tfrac{1}{2}) \left\{ \frac{2}{3} F_3(\rho,\theta) - \frac{1}{3}\frac{\cos 3\theta}{\rho^3}[-\rho \cos\theta - \tfrac{1}{3}\rho^3 \cos 3\theta + F_1(\rho,\theta)] \right.$$

$$- \frac{1}{3}\frac{\sin 3\theta}{\rho^3}[-\rho \sin\theta - \tfrac{1}{3}\rho^3 \sin 3\theta + F_2(\rho,\theta)] - \tfrac{1}{3}\rho^3 \left[-\frac{1}{\rho}\cos 2\theta + (\cos 3\theta)F_1(\rho,\theta) \right.$$

$$\left.\left. -(\sin 3\theta)F_2(\rho,\theta) \right]\right\} + \frac{1}{\pi}(\nu + \tfrac{3}{2}) \left\{ 2F_3(\rho,\theta) - \frac{1}{\rho}\cos\theta[-\rho\cos\theta + F_1(\rho,\theta)] \right.$$

$$\left. - \frac{1}{\rho}\sin\theta[-\rho\sin\theta + F_2(\rho,\theta)] - \rho[(\cos\theta)F_1(\rho,\theta) - (\sin\theta)F_2(\rho,\theta)] \right\}. \qquad (14)$$

Denoting by $(\tau_{z\rho})_{b1}$, $(\tau_{z\theta})_{b1}$ the shear stresses due to bending in the semicircular cross-section beam, and employing the relations given in Sokolnikoff (1956), we obtain:

$$(\bar{\tau}_{z\rho})_{b1} = \frac{(\tau_{z\rho})_{b1}}{\mu K_\eta R^2} = \left[\frac{\partial\Phi}{\partial\rho} + \frac{\nu}{3}(3\rho^2 \sin^3\theta - 9\rho^2 \sin\theta \cos^2\theta) + \left(\frac{8}{3\pi}\right)(1+\nu)\rho\cos 2\theta \right.$$

$$\left. -\rho^2 \sin^3\theta + \nu\rho^2 \cos 2\theta \sin\theta + \left(\frac{8}{3\pi}\right)(1+\nu)\rho \sin^2\theta \right], \qquad (15)$$

$$(\bar{\tau}_{z\theta})_{b1} = \frac{(\tau_{z\theta})_{b1}}{\mu K_\eta R^2} = \left[\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} + \frac{\nu}{3}(9\rho^2 \sin^2\theta \cos\theta - 3\rho^2 \cos^3\theta) - \left(\frac{8}{3\pi}\right)(1+\nu)\rho \sin 2\theta \right.$$

$$+ \left(\frac{16}{9\pi^2}\right)(1+\nu)\cos\theta - \rho^2(\tfrac{1}{2}\sin 2\theta \sin\theta - \nu\cos 2\theta \cos\theta)$$

$$\left. -(1+\nu)\left(\frac{16}{9\pi^2} - \frac{8}{3\pi}\rho\sin\theta\right)\cos\theta \right], \quad K_\eta = \frac{P}{EI_{\xi_1}}, \qquad (16)$$

E = Modulus of elasticity,

μ = Modulus of shear,

I_{ξ_1} = Moment of inertia of the section.

(b) *Solution for torsion of circular sector cross-section bars*

In the following, we shall derive only the solution for torsion of a quarter of a circle cross-section bar. The solution for sectors of angles $\pi/4$, $\pi/8$, etc., can be obtained similarly. According to St. Venant's theory for torsion of prismatic bars (Sokolnikoff, 1956), the equation

$$\nabla^2\Psi_2 = -2, \qquad (17)$$

must be satisfied, and the condition

$$\Psi_2 = 0 \quad \text{on the boundary} \qquad (18)$$

has to be met.

As in Naghdi (1985), we expand the right-hand side of (17) into a Fourier sine series suitable for a quarter of a circle region:

$$\frac{\partial^2\Psi_2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\Psi_2}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Psi_2}{\partial\theta^2}=-\frac{16}{\pi}\sum_{m=2,6,10}^{\infty}\frac{\sin m\theta}{m}. \tag{19}$$

The solution for (19) satisfying the condition (18) can be found in the usual way. Thus, we have

$$\Psi_2=\sum_{m=2,6,10}^{\infty}A_m^*\rho^m\sin m\theta-\left(\frac{2}{\pi}\right)\rho^2(\ln\rho-\tfrac{1}{4})\sin 2\theta$$
$$+\sum_{m=6,10,14}^{\infty}\left(-\frac{16}{\pi}\right)\rho^2\frac{1}{m(2-m)(2+m)}\sin m\theta, \tag{20}$$

in which

$$A_2^*=-\frac{1}{2\pi},$$
$$A_m^*=\left(\frac{16}{\pi}\right)\frac{1}{m(2-m)(2+m)},\quad m=6,10,14,\ldots \tag{21}$$

Employing a technique similar to the one for the summing of the series solution for Φ, we can easily write the solution Ψ_2 in closed form. For the sake of brevity, however, we shall not give the results here.

The Prandtl function Ψ_4 for the sector section of angle $\pi/4$ is obtained in the same manner:

$$\Psi_4=\sum_{m=4,12,20}^{\infty}\bar{A}_m\rho^m\sin m\theta+\sum_{m=4,12,20}^{\infty}\frac{32\rho^2\sin m\theta}{\pi m(m-2)(m+2)}, \tag{22}$$

in which

$$\bar{A}_m=\frac{-32}{\pi m(m-2)(m+2)}. \tag{23}$$

The nondimensional torsional rigidities $\bar{D}_i$ for the sector sections are defined as

$$\bar{D}_i=\int_0^{\pi/i}\int_0^1\rho^2\frac{\partial\Psi_i}{\partial\rho}\,d\rho\,d\theta,$$
$$i=2,4,8,\ldots. \tag{24}$$

Here, in (24), the index i refers to the sector cross-section of angle π/i.

(c) *Solution for the flexure of a quarter of a circle cross-section cantilever*

First, let us consider the solution for the beam which is subject to a load in its plane of symmetry. This solution and the related shear stresses can be easily obtained from Φ, $(\tau_{z\rho})_{h1}$ and $(\tau_{z\theta})_{h1}$ given previously for the semicircular beam. For example:

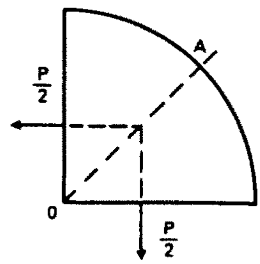

Fig. 2. Quarter of a circle cross-section beam subject to a load in its plane of symmetry.

$$(\tau'_{z\rho})_{h2} = (\tau_{z\rho})_{h1}|_{\theta} + (\tau_{z\rho})_{h1}|_{\theta+\pi/2},$$
$$(\tau'_{z\theta})_{h2} = (\tau_{z\theta})_{h1}|_{\theta} + (\tau_{z\theta})_{h1}|_{\theta+\pi/2}. \tag{25}$$

The solution obtained in this fashion is, of course, for a load $P(\sqrt{2}/2)$ (see Fig. 2).

Next, consider the general case of the quarter of a circle section beam subject to load $P/2$ (see Fig. 3). In order to obtain the solution, we follow the technique pursued by Timoshenko and Goodier (1951), and Sokolnikoff (1956) in finding the solution for the semicircular case with load perpendicular to the plane of symmetry. Looking at the first quadrant in the semicircular cross-section of the beam studied in section (a), we note that there is no shear stress in the ηZ plane. We also observe that in order to eliminate the rotation at the element located at center 0, we must apply the load $P/2$ at a distance e_2 from the origin such that

$$e_2 = \frac{M_2}{(P/2)}, \tag{26}$$

in which

$$M_2 = \int_0^{\pi/2}\int_0^R (\tau_{z\theta})_{h1} r^2 \, dr \, d\theta = \frac{PR}{2(1+\nu)\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)} \bar{M}_2, \tag{27}$$

$$\bar{M}_2 = \int_0^{\pi/2}\int_0^1 (\bar{\tau}_{z\theta})_{h1} \rho^2 \, d\rho \, d\theta. \tag{28}$$

At the same time, we must apply a twisting moment T_2 in the opposite direction of M_2 to the quarter of circle cross-section so that the rotation at the centroid shall be eliminated. The magnitude of this torque is

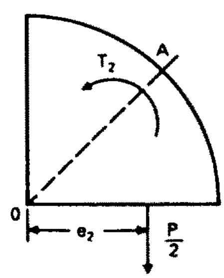

Fig. 3. Quarter of a circle cross-section beam subject to a load and a twisting moment.

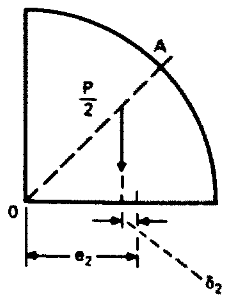

Fig. 4. Quarter of a circle cross-section beam subject to a load at its center of flexure.

$$T_2 = \frac{RP\bar{D}_2 \nu(4/3\pi)}{2(1+\nu)\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)}. \tag{29}$$

In this fashion, the total shear stress $(\tau_{z\theta})_{b2}$ for the flexure of the quarter of a circle section beam is obtained

$$\begin{aligned}(\tau_{z\theta})_{b2} &= (\tau_{z\theta})_{b1} - (\tau_{z\theta})_{T_2} \\ &= \frac{P}{2R^2(1+\nu)\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)}\left[(\bar{\tau}_{z\theta})_{b2} - \frac{4\nu}{3\pi}(\bar{\tau}_{z\theta})_{T_2}\right].\end{aligned} \tag{30}$$

Here, in (30), $(\bar{\tau}_{z\theta})_{T_2}$ represents the nondimensional shear stress produced by the twisting moment. It is given by:

$$(\bar{\tau}_{z\theta})_{T_2} = -\frac{\partial \Psi_2}{\partial \rho}. \tag{31}$$

The load at the distance e_2 from the center and the twisting moment can be combined to reduce to a single load. The distance of the line of action of this single force from the origin is found to be (see Fig. 4)

$$\begin{aligned} d_2 &= e_2 - \delta_2, \quad e_2 = R\bar{e}_2, \quad \delta_2 = R\bar{\delta}_2, \\ \bar{e}_2 &= \frac{\bar{M}_2}{(1+\nu)\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)} \cong 0.6229, \\ \bar{\delta}_2 &= \frac{\nu}{1+\nu}\left(\frac{4}{3\pi}\right)\frac{\bar{D}_2}{\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)} \cong 0.07346, \\ \nu &= 0.3 \end{aligned} \tag{32}$$

(d) *Solution for the flexure of circular sector section beams with angles $\pi/4$, $\pi/8$ etc.*

The solution for the bending of cantilevers having a circular sector cross-section with an angle of $\pi/4$ can be obtained from the solution for the symmetrical loading of a quarter-of-a-circle section beam. The procedure is exactly the same as that in the previous case. Thus, for the symmetrical loading of a beam with a $\pi/4$ angle sector (see Fig. 5), we have:

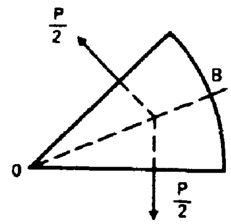

Fig. 5. One eighth of a circle cross-section beam subject to a load in its plane of symmetry.

$$(\bar{\tau}'_{z\rho})_{h4} = (\bar{\tau}'_{z\rho})_{h2}|_{\theta} + (\bar{\tau}'_{z\rho})_{h2}|_{\theta+\pi/4},$$
$$(\bar{\tau}'_{z\theta})_{h4} = (\bar{\tau}'_{z\theta})_{h2}|_{\theta} + (\bar{\tau}'_{z\theta})_{h2}|_{\theta+\pi/4}. \quad (33)$$

These shear stresses are due to the load $(2)(P/2)\sin(\pi/8)$ in the plane of symmetry of the cross-section.

For the general case of a beam with $\pi/4$ angle sector section we start from the symmetrical bending of a quarter-of-circle cross-section cantilever (see Fig. 2) and pursue the previous procedure. The quarter-of-a-circle section beam is subject to a load of $P(\sqrt{2}/4)$ in each half as shown in Fig. 6. In order to eliminate the rotation at O the force $P(\sqrt{2}/4)$ must act at a distance e_4 from the origin. Also a twisting moment T_4 has to be applied in order that the rotation at the centroid be eliminated (see Fig. 7). Similar to the case of a quarter-of-a-circle cross-section beam the force and the twisting moment can be combined to a single force $P(\sqrt{2}/4)$ at a distance $d_4 = e_4 - \delta_4$ from the origin. The expressions for torque T_4, the nondimensional distances $\bar{e}_4 = e_4/R$, $\bar{\delta}_4 = \delta_4/R$, and the shear stress $(\tau_{z\theta})_{h4}$ are

$$T_4 = \frac{PR\nu\bar{D}_4\left(\dfrac{4}{3\pi}\right)\tan\left(\dfrac{\pi}{8}\right)}{2(1+\nu)\left(\dfrac{\pi}{16}+\dfrac{1}{8}-\dfrac{8}{9\pi}\right)}, \quad (34)$$

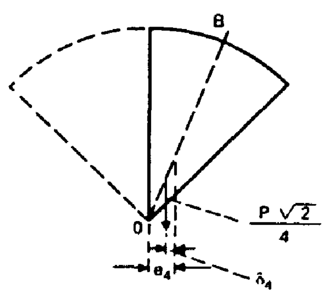

Fig. 6. One eighth of a circle cross-section beam subject to a load at its center of flexure.

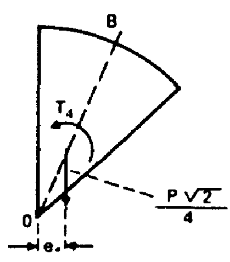

Fig. 7. One eighth of a circle cross-section beam subject to a load and a twisting moment.

$$\bar{e}_4 = \frac{4M_4}{PR\sqrt{2}} = \frac{2\bar{M}_4}{\sqrt{2}(1+\nu)\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)}$$

$$= \frac{2}{\sqrt{2}(1+\nu)\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)}\int_0^{\pi/4}\int_0^1 (\bar{\tau}'_{z\theta})_{b2}\rho^2\,\mathrm{d}\rho\,\mathrm{d}\theta = 0.1149, \quad (35)$$

$$\bar{\delta}_4 = \frac{\nu\bar{D}_4\left(\frac{4}{3\pi}\right)\tan\left(\frac{\pi}{8}\right)}{(1+\nu)\left(\frac{\pi}{16}+\frac{1}{8}-\frac{8}{9\pi}\right)} = 0.02707, \quad \bar{d}_4 = \bar{e}_4 - \bar{\delta}_4, \quad \nu = 0.3 \quad (36)$$

$$(\tau_{z\theta})_{b4} = \frac{P}{2R^2(1+\nu)\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)}\left[(\bar{\tau}'_{z\theta})_{b2} - \frac{\nu\left(\frac{4}{3\pi}\right)\tan\left(\frac{\pi}{8}\right)\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)(\bar{\tau}_{z\theta})T_4}{\left(\frac{\pi}{16}+\frac{1}{8}-\frac{8}{9\pi}\right)}\right]. \quad (37)$$

$$(\bar{\tau}_{z\theta})_{T_4} = -\frac{\partial\bar{\psi}_4}{\partial\rho}. \quad (38)$$

NUMERICAL RESULTS

Numerical results for the dimensionless shear stresses $\bar{\tau}_{z\rho}$ and $\bar{\tau}_{z\theta}$ are given for beams with circular sector cross-sections of angles π, $\pi/2$ and $\pi/4$. In Table 1, the nondimensional

Table 1. The values of nondimensional shear stress $(\bar{\tau}_{z\rho})_{b1}$ vs ρ along $\theta = \pi/2$ for the bending of the semicircular section beam

ρ	0.1	0.2	0.3	0.4	0.5
$(\bar{\tau}_{z\rho})_{b1}$	0.09099	0.16185	0.21248	0.24282	0.25283
ρ	0.6	0.7	0.8	0.9	1
$(\bar{\tau}_{z\rho})_{b1}$	0.24254	0.21201	0.16133	0.09062	0

Table 2. The values of nondimensional shear stress $(\bar{\tau}'_{z\rho})_{b2}$ vs ρ along $\theta = \pi/4$ for the symmetrical bending of a quarter-of-a-circle section cantilever

ρ	0.1	0.2	0.3	0.4	0.5
$(\bar{\tau}'_{z\rho})_{b2}$	0.09908	0.17579	0.23039	0.26313	0.27418
ρ	0.6	0.7	0.8	0.9	1
$(\bar{\tau}'_{z\rho})_{b2}$	0.26363	0.23136	0.17706	0.10018	0

Table 3. The values of nondimensional shear stress $(\bar{\tau}'_{z\rho})_{b4}$ vs ρ along $\theta = \pi/8$ for the symmetrical bending of the cantilever with circular sector section of angle $\pi/4$

ρ	0.1	0.2	0.3	0.4	0.5
$(\bar{\tau}'_{z\rho})_{b4}$	0.19826	0.35165	0.46017	0.52388	0.54293
ρ	0.6	0.7	0.8	0.9	1
$(\bar{\tau}'_{z\rho})_{b4}$	0.51773	0.44916	0.33872	0.18842	0

Table 4. The values of nondimensional shear stresses $(\bar{\tau}'_{z\rho})_{b2}$ and $(\bar{\tau}'_{z\theta})_{b2}$ vs ρ along $\theta = \pi/4$ for the general case of bending of a quarter-of-a-circle cross-section beam (see Fig. 3)

ρ	$(\bar{\tau}'_{z\rho})_{b2}$	$(\bar{\tau}'_{z\theta})_{b2}$
0.1	0.04954	0.07307
0.2	0.08789	0.12084
0.3	0.11520	0.15730
0.4	0.13156	0.18544
0.5	0.13709	0.20667
0.6	0.13181	0.22185
0.7	0.11568	0.23158
0.8	0.08853	0.23630
0.9	0.05009	0.23638

Table 5. The values of nondimensional shear stresses $(\bar{\tau}_{z\rho})_{b4}$ and $(\bar{\tau}_{z\theta})_{b4}$ vs ρ along $\theta = \pi/8$ for the general case of bending of a cantilever with circular sector section of angle $= \pi/4$ (see Fig. 7)

ρ	$(\bar{\tau}_{z\rho})_{b4}$	$(\bar{\tau}_{z\theta})_{b4}$
0.1	0.09913	0.00692
0.2	0.17582	0.01487
0.3	0.23009	0.02334
0.4	0.26194	0.03182
0.5	0.27147	0.03981
0.6	0.25887	0.04681
0.7	0.22458	0.05231
0.8	0.16936	0.05583
0.9	0.09421	0.05692

values of the shear stress $(\bar{\tau}_{z\rho})_{b1}$ vs ρ are presented along the line OA (see Fig. 1). In Tables 2 and 3, the dimensionless shear stresses $(\bar{\tau}'_{z\rho})_{b2}$ and $(\bar{\tau}'_{z\rho})_{b4}$ vs ρ are given for the symmetrical loading of beams with sector sections of angles $\pi/2$ and $\pi/4$ (see Figs 2 and 5). In Tables 4 and 5, dimensionless shear stresses $(\bar{\tau}_{z\rho})_{b2}$, $(\bar{\tau}_{z\theta})_{b2}$, $(\bar{\tau}_{z\rho})_{b4}$ and $(\bar{\tau}_{z\theta})_{b4}$ vs ρ are presented along the lines OA and OB (see Figs 3 and 7) for the general cases of bending of beams with the sector sections of $\pi/2$ and $\pi/4$ angles.

CONCLUSION

The solutions presented in this article are quite general, and can be used for beams with a semicircular section or a circular sector section of any angle. In particular, when the angles are π, $\pi/2N$ in which N is an integer, the solutions can be presented in closed forms. Since the solutions for the torsion of various sector sections are also obtained, the problems for the general cases of unsymmetrical bending, combined with torsion of cantilevers with the mentioned cross-sections are solved.

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